Structures of Relative Gromov–Witten Theory

Fenglong You

[1]University of Alberta

Dec. 07, 2020
@ Kyoto University

Outline

Outline

Moduli Space

Moduli Space of Stable Maps

$\overline{M}_{g,n}(X,d)$: the moduli space of stable maps of degree d from genus g nodal curves with n-markings to a smooth projective variety X. It consists of

$$(C, \{p_i\}_{i=1}^n) \xrightarrow{f} X,$$

where

- C is a projective, connected, nodal curve of genus g;
- $p_1, \ldots, p_n$ are distinct non-singular points of C;
- $f_*[C] = d \in H_2(X)$;
- stable: automorphisms of the maps are finite

Definition

- For each marking p_i, there is an evaluation map:

$$\mathrm{ev}_i : \overline{M}_{g,n}(X,d) \to X$$
$$\{(C, \{p_i\}_{i=1}^n) \xrightarrow{f} X\} \mapsto f(p_i).$$

Definition

- For each marking p_i, there is an evaluation map:

$$\mathrm{ev}_i : \overline{M}_{g,n}(X,d) \to X$$
$$\{(C, \{p_i\}_{i=1}^n) \xrightarrow{f} X\} \mapsto f(p_i).$$

Definition

Given cohomological classes $\gamma_i \in H^*(X)$, one can define the Gromov–Witten invariant

$$\left\langle \prod_{i=1}^n \gamma_i \right\rangle_{g,n,d}^X := \int_{[\overline{M}_{g,n}(X,d)]^{\mathrm{vir}}} \prod_{i=1}^n (\mathrm{ev}_i^* \gamma_i).$$

where $[\overline{M}_{g,n}(X,d)]^{\mathrm{vir}}$ is called the virtual fundamental class. It is served as a replacement of the fundamental class.

- If γ_i represents a submanifold of X, then invariants counts curves that go through these submanifolds at given marked points.

Dimension Constraint: the invariant is zero unless it satisfies

$$\dim := (1-g)(\dim_{\mathbb{C}} X - 3) + \int_d c_1(T_X) + n = \frac{1}{2}\sum \deg(\gamma_i).$$

Example

$$\langle\,\rangle^{Q_5}_{0,0,d} := \int_{\overline{M}_{0,0}(Q_5,d)} \mathbf{1}$$

is the "virtual" count of degree d rational curve in quintic threefold Q_5.

Outline

1. Absolute/Classical Gromov–Witten Theory
   - Definition
   - Structures
2. Relative/Logarithmic Gromov–Witten Theory
   - Definition
   - Why?
   - Structures
   - Further Results and Applications
   - Summary
   - Future Work

Structures of Absolute Gromov–Witten Theory

- String, Divisor and Dilaton equation
- Quantum cohomology
- Topological recursion relation
- Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equation
- Givental formalism: Givental's symplectic vector space, Lagrangian cone etc.
- Mirror symmetry
- Virasoro constraint
- Cohomological field theory
- Crepant/birational transformation conjecture
- Integrable system
- Modularity
- ...

Quantum Cohomology

The quantum cohomology ring $QH^*(X)$ is a deformation of the usual cohomology ring using Gromov–Witten invariants.

Quantum Product

Given $\alpha, \beta \in H^*(X)$, the quantum product is defined using three-point Gromov–Witten invariants.

$$\alpha \circ \beta = \sum_{d \in H_2^{\text{eff}}(X)} \sum_k Q^d \langle \alpha, \beta, \phi_k \rangle_{0,3,d}^X \phi^k$$

where $\{\phi_k\}$ and $\{\phi^k\}$ are dual basis of $H^*(X)$.

Remark

We can write

$$\alpha \circ \beta = \alpha \cup \beta + H.O.T.$$

where $H.O.T. = \sum_{d>0} \cdots$

Outline

Set-up

Relative Gromov–Witten theory is the enumerative theory of counting curves with tangency condition along a divisor (a codimensional one subvariety).

- X: a smooth projective variety.
- D: a smooth divisor of X.
- For $d \in H_2(X, \mathbb{Q})$, we consider a partition $\vec{k} = (k_1, \ldots, k_m)$ of $\int_d [D]$. That is,

$$\sum_{i=1}^{m} k_i = \int_d [D], \quad k_i > 0$$

- $\overline{M}_{g,\vec{k},n,d}(X, D)$: the moduli space of $(m+n)$-pointed, genus g, degree $d \in H_2(X, \mathbb{Q})$, relative stable maps to (X, D) such that the relative conditions are given by the partition $\vec{k}$.

Evaluation Maps

There are two types of evaluation maps.

$$\mathrm{ev}_i : \overline{M}_{g,\vec{k},n,d}(X, D) \to D, \quad \text{for } 1 \leq i \leq m;$$
$$\mathrm{ev}_i : \overline{M}_{g,\vec{k},n,d}(X, D) \to X, \quad \text{for } m+1 \leq i \leq m+n.$$

The first m markings are relative markings with contact order k_i, the last n markings are interior markings.

Data

- $\delta_i \in H^*(D, \mathbb{Q})$, for $1 \leq i \leq m$.
- $\gamma_{m+i} \in H^*(X, \mathbb{Q})$, for $1 \leq i \leq n$.

Definition

Definition

Relative Gromov–Witten invariants of (X, D) *are defined as*

$$\left\langle \prod_{i=1}^{m} \tau(\delta_i) \middle| \prod_{i=1}^{n} \tau(\gamma_{m+i}) \right\rangle_{g,\vec{k},n,d}^{(X,D)} := \qquad (1)$$

$$\int_{[\overline{M}_{g,\vec{k},n,d}(X,D)]^{vir}} \prod_{i=1}^{m} \mathrm{ev}_i^*(\delta_i) \prod_{i=1}^{n} \mathrm{ev}_{m+i}^*(\gamma_{m+i}).$$

Virtual Dimension Constraint

The invariant is zero unless it satisfies:

$$(1-g)(\dim_{\mathbb{C}} X - 3) + \int_d c_1(T_X) - \int_d [D] + m + n$$
$$= \frac{1}{2}\sum \deg(\gamma_i) + \frac{1}{2}\sum \deg(\delta_i)$$

Outline

Why Relative Gromov–Witten Theory?

Why Relative Gromov–Witten Theory?

Relative Gromov–Witten theory is connected with everything!

Why Relative Gromov–Witten Theory?

Relative Gromov–Witten theory is connected with everything!

- Absolute invariants and the degeneration formula.

Why Relative Gromov–Witten Theory?

Relative Gromov–Witten theory is connected with everything!

- Absolute invariants and the degeneration formula.
- Relative mirror symmetry for pairs (X, D).

Why Relative Gromov–Witten Theory?

Relative Gromov–Witten theory is connected with everything!

- Absolute invariants and the degeneration formula.
- Relative mirror symmetry for pairs (X, D).
- Logarithmic Gromov–Witten theory, a generalization of relative theory, is closely related to tropical geometry.

Why Relative Gromov–Witten Theory?

Relative Gromov–Witten theory is connected with everything!

- Absolute invariants and the degeneration formula.
- Relative mirror symmetry for pairs (X, D).
- Logarithmic Gromov–Witten theory, a generalization of relative theory, is closely related to tropical geometry.
- The tautological ring of $\overline{M}_{g,n}$ and $\overline{M}_{g,n}(D)$. Tautological classes such as Double ramification cycles.

Why Relative Gromov–Witten Theory?

Relative Gromov–Witten theory is connected with everything!

- Absolute invariants and the degeneration formula.
- Relative mirror symmetry for pairs (X, D).
- Logarithmic Gromov–Witten theory, a generalization of relative theory, is closely related to tropical geometry.
- The tautological ring of $\overline{M}_{g,n}$ and $\overline{M}_{g,n}(D)$. Tautological classes such as Double ramification cycles.
- Integrable system.

Why Relative Gromov–Witten Theory?

Relative Gromov–Witten theory is connected with everything!

- Absolute invariants and the degeneration formula.
- Relative mirror symmetry for pairs (X, D).
- Logarithmic Gromov–Witten theory, a generalization of relative theory, is closely related to tropical geometry.
- The tautological ring of $\overline{M}_{g,n}$ and $\overline{M}_{g,n}(D)$. Tautological classes such as Double ramification cycles.
- Integrable system.
- Symplectic geometry and open Gromov–Witten invariants (Which counts disks instead of curves).

Why Relative Gromov–Witten Theory?

Relative Gromov–Witten theory is connected with everything!

- Absolute invariants and the degeneration formula.
- Relative mirror symmetry for pairs (X, D).
- Logarithmic Gromov–Witten theory, a generalization of relative theory, is closely related to tropical geometry.
- The tautological ring of $\overline{M}_{g,n}$ and $\overline{M}_{g,n}(D)$. Tautological classes such as Double ramification cycles.
- Integrable system.
- Symplectic geometry and open Gromov–Witten invariants (Which counts disks instead of curves).
- Strominger–Yau–Zaslow (SYZ) mirror construction.

Why Relative Gromov–Witten Theory?

Relative Gromov–Witten theory is connected with everything!

- Absolute invariants and the degeneration formula.
- Relative mirror symmetry for pairs (X, D).
- Logarithmic Gromov–Witten theory, a generalization of relative theory, is closely related to tropical geometry.
- The tautological ring of $\overline{M}_{g,n}$ and $\overline{M}_{g,n}(D)$. Tautological classes such as Double ramification cycles.
- Integrable system.
- Symplectic geometry and open Gromov–Witten invariants (Which counts disks instead of curves).
- Strominger–Yau–Zaslow (SYZ) mirror construction.
- Gross–Siebert program: algebro-geometric version of SYZ.

Why Relative Gromov–Witten Theory?

Relative Gromov–Witten theory is connected with everything!

- Absolute invariants and the degeneration formula.
- Relative mirror symmetry for pairs (X, D).
- Logarithmic Gromov–Witten theory, a generalization of relative theory, is closely related to tropical geometry.
- The tautological ring of $\overline{M}_{g,n}$ and $\overline{M}_{g,n}(D)$. Tautological classes such as Double ramification cycles.
- Integrable system.
- Symplectic geometry and open Gromov–Witten invariants (Which counts disks instead of curves).
- Strominger–Yau–Zaslow (SYZ) mirror construction.
- Gross–Siebert program: algebro-geometric version of SYZ.
- . . . There are more reasons, but I am running out of space.

Outline

Structures of Relative Gromov–Witten Theory

Approach: using orbifolds to impose tangency conditions.

1. Consider an orbifold $X_{D,r}$, called root stack, whose orbifold structures are over D (with stablizers μ_r) and impose tangency conditions using orbifold structures.

Structures of Relative Gromov–Witten Theory

Approach: using orbifolds to impose tangency conditions.

1. Consider an orbifold $X_{D,r}$, called root stack, whose orbifold structures are over D (with stablizers μ_r) and impose tangency conditions using orbifold structures.
2. Study the relationship between relative invariants $\langle \cdots \rangle^{(X,D)}$ and (absolute) orbifold invariants $\langle \cdots \rangle^{X_{D,r}}$.

Structures of Relative Gromov–Witten Theory

Approach: using orbifolds to impose tangency conditions.

1. Consider an orbifold $X_{D,r}$, called root stack, whose orbifold structures are over D (with stablizers μ_r) and impose tangency conditions using orbifold structures.
2. Study the relationship between relative invariants $\langle \cdots \rangle^{(X,D)}$ and (absolute) orbifold invariants $\langle \cdots \rangle^{X_{D,r}}$.
3. Study structures of relative theory from structures of orbifold theory.

Structures of Relative Gromov–Witten Theory

Approach: using orbifolds to impose tangency conditions.

1. Consider an orbifold $X_{D,r}$, called root stack, whose orbifold structures are over D (with stablizers μ_r) and impose tangency conditions using orbifold structures.
2. Study the relationship between relative invariants $\langle \cdots \rangle^{(X,D)}$ and (absolute) orbifold invariants $\langle \cdots \rangle^{X_{D,r}}$.
3. Study structures of relative theory from structures of orbifold theory.

Summary

Relative theory $=$ the "limit" of orbifold theory of $X_{D,r}$ as "$r \to \infty$".

Root Stack

$X_{D,r}$: r-th root stack of X along the divisor D, where r is a positive integer. Geometrically, $X_{D,r}$ is smooth away from D and has generic stabilizer μ_r along D.

- For $d \in H_2(X, \mathbb{Q})$, we consider a partition $\vec{k} = (k_1, \ldots, k_m)$ of $\int_d [D]$. That is,
$$\sum_{i=1}^{m} k_i = \int_d [D].$$
- $\overline{M}_{g,\vec{k},n,d}(X_{D,r})$: moduli space of $(m+n)$-pointed, genus g, degree $d \in H_2(X, \mathbb{Q})$, orbifold stable maps to $X_{D,r}$ such that the orbifold conditions are given by the partition $\vec{k}$.

Evaluation Maps

Evaluation maps

$$\mathrm{ev} : \overline{M}_{g,\vec{k},n,d}(X_{D,r}) \to \underline{I}X_{D,r},$$

where $\underline{I}X_{D,r}$ is the coarse moduli space of the inertia stack $IX_{D,r}$ of $X_{D,r}$. In this case, we simply have

$$\underline{I}X_{D,r} \simeq X \cup \bigsqcup_{i=1}^{r-1} D.$$

There are $(r-1)$ copies of D are called twisted sectors. Twisted sectors are labeled by fractional numbers: $\frac{1}{r}, \frac{2}{r}, \ldots \frac{r-1}{r}$. These fractional numbers are called the **ages** of the twisted sectors.

Evaluation Maps

Evaluation maps

There are two types of evaluation maps as well.

$$\mathrm{ev}_i : \overline{M}_{g,\vec{k},n,d}(X_{D,r}) \to D, \quad \text{for } 1 \leq i \leq m;$$
$$\mathrm{ev}_i : \overline{M}_{g,\vec{k},n,d}(X_{D,r}) \to X, \quad \text{for } m+1 \leq i \leq m+n.$$

The first m markings are orbifold markings that map to twisted sectors with age k_i/r, the last n markings are interior markings.

Translation

Relative conditions translate into orbifold conditions as follows

$$\text{relative marking } \longleftrightarrow \text{ orbifold marking}$$
$$\text{contact order } k_i \longleftrightarrow \text{ age } k_i/r$$

Data (same as the data for relative invariants):

- $\delta_i \in H^*(D, \mathbb{Q})$, for $1 \leq i \leq m$.
- $\gamma_{m+i} \in H^*(X, \mathbb{Q})$, for $1 \leq i \leq n$.

Definition

Orbifold Gromov-Witten invariants of $X_{D,r}$ are defined as

$$\left\langle \prod_{i=1}^{m} \tau(\delta_i) \prod_{i=1}^{n} \tau(\gamma_{m+i}) \right\rangle_{g,\vec{k},n,d}^{X_{D,r}} := \tag{2}$$

$$\int_{[\overline{M}_{g,\vec{k},n,d}(X_{D,r})]^{vir}} \prod_{i=1}^{m} \mathrm{ev}_i^*(\delta_i) \prod_{i=1}^{n} \mathrm{ev}_{m+i}^*(\gamma_{m+i}),$$

Virtual dimension constraint

Virtual Dimension Constraint

The invariant is zero unless it satisfies:

$$(1-g)(\dim_{\mathbb{C}} X - 3) + \int_d c_1(T_X) - \int_d [D] + \frac{\int_d [D]}{r} + m + n - \sum(ages)$$
$$= \frac{1}{2}\sum \deg(\gamma_i) + \frac{1}{2}\sum \deg(\delta_i)$$

- Given the partition $\vec{k}$, $\sum(ages) = \sum k_i/r = \int_d [D]/r$. Hence (first line)=(second line) is possible, as other terms are all integers. However, $\sum(ages) = \int_d [D]/r$ is not the necessary condition. One only needs
$$\sum(ages) - \frac{\int_d [D]}{r} \in \mathbb{Z}.$$
This means that there are some orbifold invariants whose orbifold data is not exactly translated from the relative data.

Facts

Question

Will relative invariants $\langle\rangle^{(X,D)}$ and orbifold invariants $\langle\rangle^{X_{D,r}}$ be related?

Facts

Question

Will relative invariants $\langle\rangle^{(X,D)}$ and orbifold invariants $\langle\rangle^{X_{D,r}}$ be related?

Fact 1

Cadman (2006): the moduli space $\overline{M}_{g,\vec{k},n,d}(X_{D,r})$ of orbifold stable maps provides an alternative compactification of the space of relative stable maps.

Facts

Question

Will relative invariants $\langle\rangle^{(X,D)}$ and orbifold invariants $\langle\rangle^{X_{D,r}}$ be related?

Fact 1

Cadman (2006): the moduli space $\overline{M}_{g,\vec{k},n,d}(X_{D,r})$ of orbifold stable maps provides an alternative compactification of the space of relative stable maps.

Fact 2

- Maulik-Pandharipande (2006): $\langle\rangle^{(X,D)}$ is determined by $\langle\rangle^{X}$, $\langle\rangle^{D}$ and the restriction map $H^*(X) \to H^*(D)$.
- Tseng-Y(2016): $\langle\rangle^{X_{D,r}}$ is also determined by $\langle\rangle^{X}$, $\langle\rangle^{D}$ and the restriction map $H^*(X) \to H^*(D)$ (of course, plus the dependence on r as well).

Facts

Fact 3

The anticanonical class for root stack is

$$-K_{X_{D,r}} = -K_X - D + D/r.$$

It approaches to $-K_X - D$ as r goes to ∞.

Facts

Fact 3

The anticanonical class for root stack is

$$-K_{X_{D,r}} = -K_X - D + D/r.$$

It approaches to $-K_X - D$ as r goes to ∞.

Implication:

These facts indicate that relative and orbifold invariants may be related when r is sufficiently large.

Relation between relative and orbifold theory in genus zero

Theorem (Abramovich–Cadman–Wise, 2017 (arXiv 2010))

Genus zero:

$$\langle \cdots \rangle^{(X,D)} = \langle \cdots \rangle^{X_{D,r}},$$

for $r \gg 1$. That is, orbifold invariants stabilized as $r \to \infty$ and relative invariants are limits of orbifold invariants.

Relation between relative and orbifold theory in genus zero

Theorem (Abramovich–Cadman–Wise, 2017 (arXiv 2010))

Genus zero:

$$\langle \dots \rangle^{(X,D)} = \langle \dots \rangle^{X_{D,r}},$$

for $r \gg 1$. That is, orbifold invariants stabilized as $r \to \infty$ and relative invariants are limits of orbifold invariants.

Remark

The equality includes all the genus zero relative invariants but NOT ALL the genus zero orbifold invariants of $X_{D,r}$. Recall that contact orders k_i are translated to ages k_i/r. When $r \gg 1$, all the ages in the above formula are **small**. *However, there can also be orbifold invariants with* **large** *ages:* $\frac{r-1}{r}, \frac{r-2}{r}, \dots$ *etc; mid-ages: 1/2,1/3, 1/4 etc.*

Counterexample in Genus 1

Counterexample in Genus 1, given by D. Maulik.

- Let $X = E \times \mathbb{P}^1$, where E is an elliptic curve.
- Let $D = D_0 \cup D_\infty$, where $D_0 = E_0$, $D_\infty = E_\infty$.
- $[f] \in H_2(E)$.
- Relative invariant $\langle\rangle^{(X,D)}_{1,0,[f]} = 0$.
- Orbifold invariant $\langle\rangle^{X_{(D_0,r),(D_\infty,s)}}_{1,0,[f]} = r + s$.

Counterexample in Genus 1

Counterexample in Genus 1, given by D. Maulik.

- Let $X = E \times \mathbb{P}^1$, where E is an elliptic curve.
- Let $D = D_0 \cup D_\infty$, where $D_0 = E_0$, $D_\infty = E_\infty$.
- $[f] \in H_2(E)$.
- Relative invariant $\langle \rangle^{(X,D)}_{1,0,[f]} = 0$.
- Orbifold invariant $\langle \rangle^{X_{(D_0,r),(D_\infty,s)}}_{1,0,[f]} = r + s$.

The equality does not hold in higher genus. It is the end of one story, but it is also the begining of another story. There are some natural questions that we can ask.

Questions

Question 1

What is the relation in higher genus?

Question 2

Do those extra orbifold invariants also stabilize? What is the relation with relative invariants?

Question 3

Is it a version of relative mirror symmetry for pairs (X, D)?

Questions

Question 1

What is the relation in higher genus? (Tseng–Y, April 2018, June 2018, Jan. 2020)

Question 2

Do orbifold invariants with large ages stabilize? What is the relation with relative invariants? (Fan–Wu–Y, Oct. 2018, July 2019)

Question 3

Is it a version of relative mirror symmetry for pairs (X, D)? (Fan–Tseng–Y, Nov. 2018)

Question 1

Question 1

What is the relation in higher genus?

Question 1

Question 1

What is the relation in higher genus?

Question:

Can we find a conjectural relation by looking at the counterexample of D. Maulik?

- Relative invariant $\langle\rangle^{(X,D)}_{1,0,[f]} = 0$.
- Orbifold invariant $\langle\rangle^{X_{(D_0,r),(D_\infty,s)}}_{1,0,[f]} = r + s$.

Question 1

Question 1

What is the relation in higher genus?

Question:

Can we find a conjectural relation by looking at the counterexample of D. Maulik?

- Relative invariant $\langle\rangle_{1,0,[f]}^{(X,D)} = 0$.
- Orbifold invariant $\langle\rangle_{1,0,[f]}^{X_{(D_0,r),(D_\infty,s)}} = r + s$.

- True story: there was someone who can see the correct relation in just four seconds after seeing the counterexample. However, it took us four years to finally see the relation.

Question 1

Hint:

In general, $\langle \cdots \rangle^{X_{D,r}}$ is a function in r. In genus zero, it becomes a constant function when $r \gg 1$. **What is the simplest non-constant function?**

Question 1

Hint:

In general, $\langle \cdots \rangle^{X_{D,r}}$ is a function in r. In genus zero, it becomes a constant function when $r \gg 1$. **What is the simplest non-constant function?**

Answer:

A **polynomial!** (A linear function in genus 1, [Tseng–Y, Jan. 2020].)

Question 1

Hint:

In general, $\langle \cdots \rangle^{X_{D,r}}$ is a function in r. In genus zero, it becomes a constant function when $r \gg 1$. **What is the simplest non-constant function?**

Answer:

A **polynomial!** (A linear function in genus 1, [Tseng–Y, Jan. 2020].)

Theorem (Tseng–Y, April 2018, June 2018)

For higher genus invariants

$$\langle \cdots \rangle^{(X,D)} = \left[\langle \cdots \rangle^{X_{D,r}} \right]_{r^0}, r \gg 1,$$

where the orbifold invariant $\langle \cdots \rangle^{X_{D,r}}$ *is a polynomial in* r *for* $r \gg 1$ *and* $[\cdots]_{r^0}$ *means the constant term of the polynomial.*

Question 1

Remark

Our approach is different from [ACW]'s, but we are also able to give another proof of the exact equality in genus zero. One advantage of our approach is that it is easy to see the difference between genus zero and higher genus comparison (although it is not easy to see it just from my slides).

Question 1

Remark

Our approach is different from [ACW]'s, but we are also able to give another proof of the exact equality in genus zero. One advantage of our approach is that it is easy to see the difference between genus zero and higher genus comparison (although it is not easy to see it just from my slides).

Remark

We use degeneration, virtual localization techniques as well as a (highly nontrivial) polynomiality proved by Janda–Pandharipande–Pixton–Zvonkine (2017, 2018). It also explains why we were not able to do it back in 2014.

Question 2: in Genus Zero

Question 2

Do orbifold invariants with large ages stabilize? What is the relation with relative invariants?

Let us first focus on the GENUS ZERO invariants.

Question 2: in Genus Zero

Question 2

Do orbifold invariants with large ages stabilize? What is the relation with relative invariants?

Let us first focus on the GENUS ZERO invariants.
The answer is No!

Question 2: in Genus Zero

Question 2

Do orbifold invariants with large ages stabilize? What is the relation with relative invariants?

Let us first focus on the GENUS ZERO invariants.

The answer is No!

But also Yes!

Question 2: in Genus Zero

Question 2

Do orbifold invariants with large ages stabilize? What is the relation with relative invariants?

Let us first focus on the GENUS ZERO invariants.

The answer is No!

But also Yes!

- No: $\langle\rangle^{X_{D,r}}$ depends on r as $r \to \infty$.

Question 2: in Genus Zero

Question 2

Do orbifold invariants with large ages stabilize? What is the relation with relative invariants?

Let us first focus on the GENUS ZERO invariants.

The answer is No!

But also Yes!

- No: $\langle\rangle^{X_{D,r}}$ depends on r as $r \to \infty$.
- Yes: $\langle\rangle^{X_{D,r}}$ goes to 0 as $r \to \infty$.

Question 2: in Genus Zero

Question 2

Do orbifold invariants with large ages stabilize? What is the relation with relative invariants?

Let us first focus on the GENUS ZERO invariants.

The answer is No!

But also Yes!

- No: $\langle\rangle^{X_{D,r}}$ depends on r as $r \to \infty$.
- Yes: $\langle\rangle^{X_{D,r}}$ goes to 0 as $r \to \infty$.

However, 0 is not a really interesting answer to the question. The interesting answer is the following.

Question 2: in Genus Zero

Theorem (Fan-Wu-Y, Oct. 2018)

Let m_- be the number of large ages markings, then $r^{m_-}\langle\rangle^{X_{D,r}}$ stabilized as $r \gg 1$ (and the limit is not zero).

Question 2: in Genus Zero

Theorem (Fan-Wu-Y, Oct. 2018)

Let m_- be the number of large ages markings, then $r^{m_-}\langle\rangle^{X_{D,r}}$ stabilized as $r \gg 1$ (and the limit is not zero).

Question:

What are the limits of such orbifold invariants? What is the relation with relative invariants? Recall that the genus zero result of [ACW] has already covered all relative invariants.

Question 2: in Genus Zero

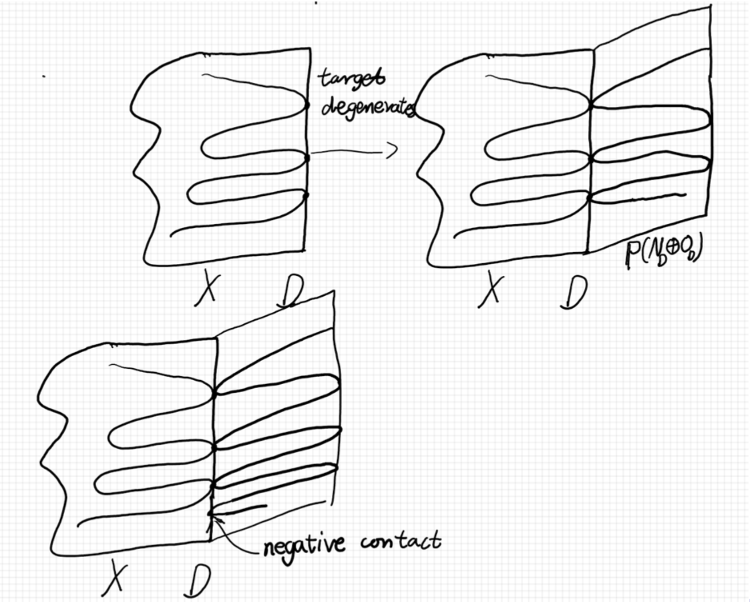
Theorem (Fan–Wu–Y, Oct. 2018)

Let m_- be the number of large ages markings, then

$$r^{m_-}\langle\rangle^{X_{D,r}} = \langle\rangle^{(X,D)}, r \gg 1$$

where $\langle\rangle^{(X,D)}$ in here are relative invariants with **negative** *contact orders defined in [Fan-Wu-Y, Oct 2018].*

Question 2: in Genus Zero



Question 2: in Genus Zero

Recall that, in the definition of relative invariants, the relative conditions (contact orders) need to satisfy

$$\sum k_i = \int_d [D], k_i > 0.$$

relative conditions translate into orbifold conditions as follows

$$\text{positive contact order } k_i \longleftrightarrow \text{small age } k_i/r$$

orbifold conditions need to satisfy

$$\sum ages - \frac{\int_d [D]}{r} \in \mathbb{Z}.$$

If we do not require contact orders k_i to be positive, then

$$\text{negative contact order } k_i \longleftrightarrow \text{large age } (r + k_i)/r$$

Structure of relative Gromov–Witten theory

Relative Gromov–Witten invariants with (possibly) negative contact orders have the following properties:

- Relative quantum cohomology ring
- Topological recursion relation
- WDVV equation
- Givental's formalism: Givental's symplectic vector space, Lagrangian cone etc.
- Virasoro constraint (in genus zero).

While relative Gromov–Witten theory without negative contact orders does not satisfy these properties.

Structure of relative Gromov–Witten theory

Relative Gromov–Witten invariants with (possibly) negative contact orders have the following properties:

- Relative quantum cohomology ring
- Topological recursion relation
- WDVV equation
- Givental's formalism: Givental's symplectic vector space, Lagrangian cone etc.
- Virasoro constraint (in genus zero).

While relative Gromov–Witten theory without negative contact orders does not satisfy these properties.

Remark

For those who are familiar with Gross–Siebert program, similar invariants also appear in their program where they are called punctured Gromov–Witten invariants.

Question 2: Higher Genus

Theorem (Fan–Wu–Y, July 2019)

For higher genus invariants,

$$\left[\langle\cdots\rangle^{X_{D,r}}\right]_{r^0}=\langle\cdots\rangle^{(X,D)}, r\gg 1$$

after multiplying by suitable power of r, where $\langle\cdots\rangle^{(X,D)}$ *in here are higher genus relative invariants with* **negative** *contact orders defined in [Fan-Wu-Y, July 2019].*

Question 2: Higher Genus

Theorem (Fan–Wu–Y, July 2019)

For higher genus invariants,

$$\left[\langle \cdots \rangle^{X_{D,r}}\right]_{r^0} = \langle \cdots \rangle^{(X,D)}, r \gg 1$$

after multiplying by suitable power of r, where $\langle \cdots \rangle^{(X,D)}$ in here are higher genus relative invariants with **negative** *contact orders defined in [Fan-Wu-Y, July 2019].*

Partial CohFT

As a result, relative Gromov–Witten theory forms a partial cohomological field theory (CohFT) which is a CohFT without the loop axiom.

Question 3

Question 3

Is it a mirror symmetry statement for pairs (X, D)?

It is a matter of finding the I-function.

Theorem (Fan–Tseng–Y, Nov 2018)

The I-function for the root stack $X_{D,r}$ is

$$I_{X_{D,r}}(Q,t,z) = \sum_{d\in\overline{\mathrm{NE}}(X)} J_{X,d}(t,z)Q^d \left(\frac{\prod_{0<a\leq D\cdot d}(D+az)}{\prod_{\langle a\rangle=\langle D_r\cdot d\rangle, 0<a\leq D_r\cdot d}(D_r+az)}\right) 1_{\langle -D_r\cdot d\rangle}.$$

It is proved by using orbifold quantum Lefschetz (Tseng, 2010) and mirror theorem for toric stack bundles (Jiang–Tseng–Y, 2017).

Question 3

Taking (a suitable) limit as $r \to \infty$, we obtain

Theorem (Fan–Tseng–Y, Nov 2018)

The I-function for the pair (X, D) is

$$I_{(X,D)}(Q,t,z) = \sum_{d \in \overline{\mathrm{NE}}(X)} J_{X,d}(t,z) Q^d \left(\prod_{0 < a \leq D \cdot d - 1} (D + az) \right) [1]_{-D \cdot d}.$$

Outline

Further Results

1. Polynomiality of $\langle \cdots \rangle_g^{X_{D,r}}$: Tseng-Y, Jan. 2020.

- The degree is bounded by $2g-1$ when the genus $g \geq 1$.
- Other coefficients of the polynomial are lower genus relative invariants.

2. Towards a modified CohFT for relative Gromov–Witten theory: Y, March 2020.

3. Mirror symmetry for the pair (X, D) when D is a simple normal crossing divisor: Tseng-Y, June 2020.

4. A new Gromov–Witten theory for the pair (X, D) when D is a simple normal crossing divisor: Tseng-Y, August, 2020.

Applications

1. Relative invariants=open invariants=instanton corrections of SYZ mirror for toric Calabi–Yau orbifolds: Y, May 2019.

2. Two "relative periods" can be "glued" to an (absolute) period (Doran–Harder–Thompson Conjecture): Doran–Kostiuk–Y, Oct. 2019.

3. Local invariants=log invariants=orbifold invariants (Local-log-orbifold principle): Tseng-Y, June 2020.

4. Quantum periods=classical periods in the Fano search program: Tseng–Y, June 2020.

5. The Frobenius structure conjecture and intrinsic mirror symmetry in the Gross–Siebert program: Tseng–Y, Aug. 2020.

Application: The Doran–Harder–Thompson conjecture

Let X be a Calabi–Yau manifold. Consider the Tyurin degeneration of X

$$X \rightsquigarrow X_1 \cup_D X_2.$$

For example, degeneration of quintic

$$Q_5 \rightsquigarrow \mathrm{Bl}_C \mathbb{P}^3 \cup_{K3} Q_4.$$

The mirrors for (X_1, D) and (X_2, D) are LG models $W_1 : X_1^\vee \to \mathbb{C}$ and $W_2 : X_2^\vee \to \mathbb{C}$ respectively, where the generic fibers of W_1 and W_2 are $D^\vee$, mirror to D.

Conjecture (Doran–Harder–Thompson, 2015)

The LG models for (X_1, D) and (X_2, D) can be glued together to $X^\vee \to \mathbb{P}^1$, where $X^\vee$ is mirror to X.

Application: Mirror symmetry and degeneration

Theorem (Doran–Kostiuk–Y, Oct. 2019)

The DHT conjecture holds for toric complete intersections at the level of periods. The following gluing formula holds for holomorphic periods:

$$f_0^X(q) \star_q f_0^D(q) = \frac{1}{2\pi i} \oint f_0^{X_1}(q,y) \star_q f_0^{X_2}(q.y) \frac{dy}{y},$$

where $\star_q$ is the Hadamard product: $(\sum a_n q^n) \star_q (\sum b_n q^n) = \sum a_n b_n q^n$. Holomorphic periods are solutions to Picard–Fuchs equations. The gluing formula also holds for the basis of solutions (I-functions) for the Picard–Fuchs equations.

- Periods for the mirror of $X \longleftrightarrow \mathrm{GW}(X)$.
- Periods for the mirror of $D \longleftrightarrow \mathrm{GW}(D)$.
- Periods for the mirror of $(X_1, D) \longleftrightarrow \mathrm{GW}(X_1, D)$.
- Periods for the mirror of $(X_2, D) \longleftrightarrow \mathrm{GW}(X_2, D)$.

Outline

Summary

- Relative invariants are limits of orbifold invariants.
- Relative invariants with negative contact orders are **inevitable** if we want to have a good structure of relative theory.
- Mirror symmetry holds for relative pairs.
- Several further results and applications to enumerative geometry and mirror symmetry.

Outline

Future Work

An (incomplete) list:

- Virasoro constraint
- Relation with punctured logarithmic Gromov–Witten theory and tropical geometry
- Further relation with Gross–Siebert program
- Further relation with SYZ mirror symmetry and open Gromov–Witten invariants
- Compatibility between the A-model degeneration formula and the B-model gluing formula
- DHT for more general Calabi–Yau manifolds
- DHT on the categorical level (derived category)
- Topological recursion
- Integrable system
- $\cdots$

Reference I

D. Abramovich, C. Cadman, J. Wise, *Relative and orbifold Gromov-Witten invariants,* Algebr. Geom. 4 (2017), no. 4, 472–500.

C. Doran, J. Kostiuk and F. You, *The Doran–Harder–Thompson Conjecture for toric complete intersections,* arXiv:1910.11955.

. Fan, H.-H. Tseng and F. You. *M*irror theorems for root stacks and relative pairs. Selecta Math. (N.S.) 25 (2019), no. 4, Paper No. 54. arXiv:1811.10807

. Fan, L. Wu and F. You. *S*tructures in genus-zero relative Gromov–Witten theory. J. Topol. 13 (2020), no. 1, 269–307. arXiv:1810.06952.

H. Fan, L. Wu, F. You, *Higher genus relative Gromov–Witten theory and double ramification cycles,* arXiv:1907.07133.

Reference II

- H.-H. Tseng, F. You, *On orbifold Gromov-Witten theory in codimension one,* J. Pure Appl. Algebra 220 (2016), no. 10, 3567–3571.
- H.-H. Tseng, F. You, *Higher genus relative and orbifold Gromov-Witten invariants of curves,* arXiv:1804.09905.
- H.-H. Tseng, F. You, *Higher genus relative and orbifold Gromov-Witten invariants,* To appear in Geom. Topol. arXiv:1806.11082.
- H.-H. Tseng, F. You, *On the polynomiality of orbifold Gromov–Witten theory of root stacks,* arXiv:2001.00244.
- H.-H. Tseng, F. You, *A mirror theorem for multi-root stacks and applications,* arXiv:2006.08991.

Reference III

H.-H. Tseng, F. You, *A Gromov-Witten theory for simple normal-crossing pairs without log geometry,* arXiv:2008.04844.

F. You, *Relative Gromov–Witten invariants and the enumerative meaning of mirror maps for toric Calabi–Yau orbifolds,* To appear in Trans. Amer. Math. Soc. arXiv:1905.03885.

F. You, *Gromov–Witten invariants of root stacks with mid-ages and the loop axiom,* arXiv:2003.09838.

The End

Thank you!